The Kadanoff lower-bound variational renormalization group applied to an SU(2) lattice spin model

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Abstract. We apply the variational lower-bound renormalization group transformation of Kadanoff to an SU(2) lattice spin model in two and three dimensions. Even in the one-hypercube framework of this renormalization group transformation the present model is characterized by having an infinite basis of fundamental operators. We investigate whether the lower-bound variational renormalization group transformation yields results stable under different truncations of this operator basis. Our results show that for this particular spin model this is not the case.

1. Introduction

The variational lower-bound renormalization group transformation (LBRG) introduced by Kadanoff [1, 2] is known to give surprisingly accurate results for the critical behaviour of almost all spin models on which it has been applied (see e.g. [3] for a rather exhaustive review of the results obtained until around 1982). In the case of discrete-spin models for which comparison can be made with exact solutions, the agreement is typically in the range of 1-2%. There are, however, also known problems with the method. Indeed, attempting to improve the LBRG by a more consistent choice of the variational parameter of the transformation leads to worse agreement with exactly known answers [4]. Still, overall the results obtained by the LBRG are very impressive. Recently the method has even been extended to systems with local gauge invariance and, for some discrete gauge groups at least, the technique yields results accurate to within a few per cent [5].

One very appealing feature of the LBRG is that it automatically becomes a one-hypercube approximation. For discrete models the basis of operators is then finite, and one obtains a closed set of recursion relations among a finite (typically quite small) set of coupling constants.

One obvious question is how the LBRG performs if the basis of operators, even on just one hypercube, is infinite. We know of at least one such model which has been studied in this approximation, the two-dimensional XY model [6]†. This is, however, an extremely difficult model to analyse from a local real-space renormalization group

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† Another interesting model with a continuous variable, somewhat similar to the model under study here, has been considered in [7]. However, in that case a projection down to a discrete Z(2) invariant Ising-like model was made as an initial step. Thus the LBRG could proceed as usual, once this mapping had been performed.

point of view, since the infinite-order phase transition there is believed to be driven by interactions among extended objects—vortices. In fact, the LBRG instead gives (in the approximation of [6]) a usual second-order phase transition with ordinary critical exponents. However, the 2D XY model is indeed very special (an ordinary second-order phase transition is forbidden by the Mermin–Wagner theorem since the symmetry that gets broken is U(1), i.e. continuous) and cannot be viewed as a typical spin model with an infinite set of couplings.

In this paper we shall describe the results of applying the LBRG to another continuous spin model; an effective Polyakov line action of finite temperature SU(2) lattice gauge theory [8]. The partition function of this model is given by

$$\mathscr{Z} = \int [\mathrm{d} W] \exp[\mathscr{S}_{\mathrm{eff}}(W)] \tag{1.1}$$

where $\mathscr{S}_{\mathrm{eff}}(W)$ is an effective action of nearest-neighbour couplings

$$\mathscr{S}_{\mathrm{eff}}(W) = \tfrac{1}{2}J \sum_{x,j} \{\mathrm{Tr}(W(x))\,\mathrm{Tr}(W^{\dagger}(x+j)) + \mathrm{Tr}(W^{\dagger}(x))\,\mathrm{Tr}(W(x+j))\} \tag{1.2}$$

and the partition-function integral is performed over the SU(2) Haar measure. Although we shall not be concerned here with the original motivation for studying this model (it describes the physics of finite temperature deconfinement phase transitions for, in general, SU(N) gauge theories), we shall only point out that it can be derived by a strong coupling character expansion of the lattice-regularized SU(2) (or, in general, SU(N)) gauge theory of action

$$\mathscr{S} = \tfrac{1}{4}\beta \sum_{\mathrm{plaq}} (\mathrm{Tr}\, U_p + \mathrm{Tr}\, U_p^{\dagger}). \tag{1.3}$$

Here the sum runs over elementary plaquettes on a d-dimensional lattice, and the (oriented) plaquette (in the μ-ν plane) is given by

$$U_p = U_{\mu}(x)\,U_{\nu}(x+\mu)\,U_{\mu}^{\dagger}(x+\mu+\nu)\,U_{\nu}^{\dagger}(x+\nu) \tag{1.4}$$

in a hopefully obvious notation.

The partition function of this lattice gauge theory is given by the integral of $\exp\{\mathscr{S}\}$ over the Haar measure of all link variables U_{μ}. Finite temperature in this Euclidean quantum field theory is imposed by requiring periodicity in the temporal direction; with unit lattice spacing we simply have $T = 1/N_{\tau}$, where N_{τ} equals the number of links in the time direction.

With this notation, the coupling constant of the effective action is given by

$$J = \left[\frac{I_2(\beta)}{I_1(\beta)}\right]^{N_{\tau}} \tag{1.5}$$

where $I_n(x)$ is the modified Bessel function of order n, and

$$W(x) = \prod_{\tau=1}^{N_{\tau}} U_0(x, \tau) \tag{1.6}$$

but we shall not in the following refer back to the origin of this effective lattice theory. However, we shall always keep in mind that we are dealing with, in this particular case, SU(2)-valued 'spin' variables.

At a possible phase transition it is not the local SU(2) invariance, but rather a global Z(2) symmetry, that gets spontaneously broken [9]. That the effective action

(1.2) possesses such a Z(2) invariance is clear since $W(x) \rightarrow -W(x)$ for all x obviously is a symmetry of the action. It is also a symmetry of the measure since $Z(2) \subset SU(2)$, and the Haar measure is invariant under both left and right $SU(2)$-valued multiplication. It follows that $\mathrm{Tr}\, W(x)$ is a good order parameter for this $Z(2)$-breaking phase transition.

Ideas of universality have been put forward to understand these deconfinement phase transitions [10] of $(d+1)$-dimensional $SU(N)$ lattice gauge theories in terms of d-dimensional $Z(N)$ spin systems with short-range spin–spin interactions. The effective Polyakov line action (1.2) can be viewed as one explicit step in this direction. Still, although the action [1.2] for $SU(2)$ resembles an Ising model (albeit with $SU(2)$-valued 'spins'), the Boltzmann weight is integrated with the full $SU(2)$ Haar measure. It is therefore *a priori* not completely obvious that even if the model (1.2) has a second-order phase transition, it should belong to the Ising fixed point. This originally served as our motivation for trying to understand this model within an accurate analytical renormalization group scheme, such as the LBRG. But first we had to investigate to what extent, if at all, the LBRG is a useful framework for such a theory, which has an infinite operator basis on just one hypercube. This paper is an account of that study.

2. The lower-bound renormalization group transformation

Given a Hamiltonian $\mathcal{H}(x)$, any exact renormalization group is based on forming an iterative sequence $\mathcal{H}(x) \rightarrow \mathcal{H}'(x) \rightarrow \mathcal{H}''(x) \rightarrow \ldots \rightarrow \mathcal{H}^{(n)}(x)$ through transformations of the form

$$ e^{\mathcal{H}'(x')} = \int [\mathrm{d}x] \mathcal{P}(x', x)\, e^{\mathcal{H}(x)} \tag{2.1} $$

where conservation of the total free energy (or the partition function) requires a normalization condition on the otherwise as yet unspecified projection operator $\mathcal{P}(x', x)$:

$$ \int [\mathrm{d}x'] \mathcal{P}(x', x) = 1. \tag{2.2} $$

Kadanoff's LBRG [1, 2] is an explicit, but only approximate, realization of such a program. It is based on two main ingredients: bond moving, and an optimized choice of a variational parameter (easily generalizable to more than one variational parameter). Bond moving can be described most easily here by replacing the original Hamiltonian $\mathcal{H}(x)$ by $\mathcal{H}(x) + \delta \mathcal{V}(x)$ in such a way that all interactions fall on independent hypercubes, i.e. such that the sum (integral) in (2.1) typically can be performed explicitly. Since bond moving implies $\langle \delta \mathcal{V}(x) \rangle = 0$ (where expectation values are taken relative to the original Hamiltonian), this is easily seen to give a lower bound on the actual free energy. The variational parameter(s) is then simply used to optimize this lower bound.

In the original definition of the LBRG [1, 2] for the two-dimensional Ising model, this scheme was actually carried out in two steps. First, an *exact* RG transformation, in the form of a decimation in which half of all original spins were integrated out [12], was applied to the starting Hamiltonian. This has the advantage of directly mapping us onto a Hamiltonian which is symmetric under an arbitrary permutation of spins in

each hypercube. We call this the *symmetric* LBRG. As emphasized by Burkhardt [11], this first operation is not necessary; one can equally well start with a Hamiltonian without this permutation symmetry. In the case of the Ising model this gives a slightly different result. We shall here consider both types of transformations. As the symmetric LBRG is by far the simplest, we shall start with that†.

First we rewrite the action (1.1) in terms of the diagonalized SU(2) element

$$Z(x) \equiv \text{Tr } W(x) = \text{Tr}\begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix} = 2\cos(\varphi/2) \tag{2.3}$$

for $0 \le \varphi \le 4\pi$. By relabelling $K_{nn} = J$, the 'Hamiltonian' of the model then takes a more familiar-looking form

$$\mathcal{H}(Z) = K_{nn} \sum_{\langle ij \rangle} Z_i Z_j \tag{2.4}$$

where the subscripts simply refer to positions x_i on the lattice.

As mentioned above, we begin by performing a spin decimation using the Hamiltonian (2.4). We denote by W the spins that are integrated out, and by Z the spins that remain (see figure 1). Then every spin W interacts with 2^d nearest-neighbour spins Z, and the decimation transformation for one of those spins W can be written as
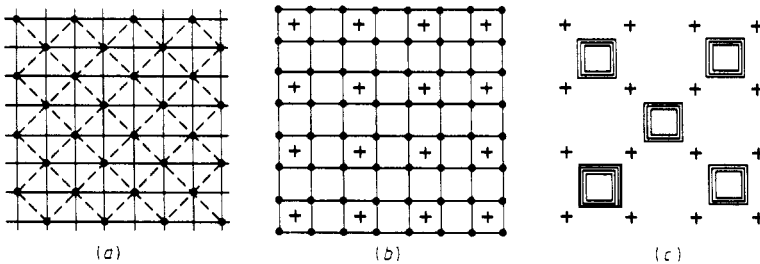
$$e^{\mathcal{A}(Z)} = \int [\text{d}W] \exp\left(K_{nn} W \sum_{i=1}^{2^d} Z_i \right) \tag{2.5}$$

where $[\text{d}W]$ is the usual SU(2) Haar measure, written in terms of the new variables as

$$[\text{d}W] = \frac{\text{d}W}{\pi} \sqrt{1 - \tfrac{1}{4} W^2} = \frac{\text{d}\varphi}{2\pi} \sin^2(\varphi/2). \tag{2.6}$$

The integral (1.6) is straightforwardly evaluated by means of the following identity involving modified Bessel functions of order n, $I_n(x)$:

$$e^{A\cos\varphi} = \sum_{n=-\infty}^{\infty} I_n(A) e^{in\varphi} \tag{2.7}$$

Figure 1. The decimation and bond moving in two dimensions. The dots denote the Z-spins and the crosses W-spins. (*a*) Shows the decimation in which the lattice is scaled by a factor $\sqrt{2}$. (*b*) Shows how the W-spin lattice is embedded in the Z-spin lattice and (*c*) shows how the bond moving transfers all the interactions into independent hypercubes.

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† Note that in order for the decimation transformation to map us onto a hypercubic lattice, we need in general to start with the original model on a body-centred hypercubic lattice. In the case of $d = 2$ this turns out to be equivalent to a rotated hypercubic lattice of different spacing, see figure 1.

and the result is

$$e^{\hbar(Z)} = I_0\left(2K_{nn} \sum_{i=1}^{2^d} Z_i\right) - I_2\left(2K_{nn} \sum_{i=1}^{2^d} Z_i\right) \tag{2.8}$$

up to an overall normalization constant, which of course turns out to be irrelevant.

The new Hamiltonian $\hbar(Z)$ now describes the interactions between trivially renormalized spins, Z, of larger lattice spacing. The full renormalized Hamiltonian is of course found by summing the Hamiltonian $\hbar(Z)$ over all fundamental hypercubes of 2^d new spins Z_i. As there are an infinite number of ways these 2^d spins can interact among themselves, we must already at this point expand $\hbar(Z)$ on an infinite basis of operators. Let us write

$$\hbar(Z) = \sum_{n=0}^{\infty} \hbar_n(Z) \tag{2.9}$$

where $\hbar_n(Z)$ is the part of the decimation Hamiltonian that can be built by interactions of n spins. In our case, where the 'spins' are traces of elements of SU(2), self-interactions are also generated. A general n-spin interaction can hence be written as

$$K_{nx} \sum_{i=1}^{2^d} Z_i^{k_i} \qquad \text{where} \qquad \sum_{i=1}^{2^d} k_i = n. \tag{2.10}$$

We determine the decimation relation for each coupling constant K_{nx} from (2.8) by an ordinary Taylor expansion on the above basis. Thus, if we denote the right-hand side of (2.8) by $\mathcal{J}(Z, K_{nn})$, this gives us

$$K_{nx} = \prod_{i=1}^{2^d} \frac{1}{k_i!} \frac{\partial^{k_i}}{\partial Z_i^{k_i}} \ln \mathcal{J}(Z, K_{nn}) \Big|_{Z_i=0} \tag{2.11}$$

The above relation gives all K_{nx}, up to any given order n. But of course we have no *a priori* knowledge of how fast it converges.

After the decimation we have on each hypercube 2^d spins Z_i that interact together as dictated by $\hbar(Z)$ and the above expansions. But if we look at the interactions between two particular spins, they can of course belong to several different hypercubes. The idea of bond moving is to move all interactions in such a way that all interactions lie within isolated hypercubes, so that no two hypercubes share interactions (see again figure 1). Combining this step with an RG projection operator of the form

$$\mathcal{P}(W, Z) \equiv \exp\left(p \sum_{i=1}^{2^d} Z_i W_i\right) \tag{2.12}$$

where p is a free variational parameter, one finally ends up with the LBRG transformation

$$e^{[\hbar'(W)]} = \int \prod_{i=1}^{2^d} [dZ_i] \exp\left(2^d \hbar(Z) - u(Z, p) + p \sum_{i=1}^{2^d} Z_i W_i\right) \tag{2.13}$$

where the function $u(Z, p)$ is determined by the normalization condition on $\mathcal{P}(Z, W)$:

$$e^{u(Z,p)} = \int [dW] \exp\left(pW \sum_{i=1}^{2^d} Z_i\right). \tag{2.14}$$

The relation (2.13) gives us the relation between the starting Hamiltonian $\hbar(Z)$, and the renormalized Hamiltonian $\hbar'(W)$. To turn this into a working iterative scheme that can help us determine the fixed point properties of the system under study, we

must again expand the new Hamiltonian on a given basis; this time, of course, for convenience chosen in accordance with the previous decimation transformation. Note that, in fact, the normalization condition (2.14) is identical in form to the integral encountered at the decimation. Hence the result is the same, and the LBRG can be written as

$$\hbar'(W) = 2^d K_0 + \ln \int \prod_{i=1}^{2^d} [\mathrm{d}Z_i] \mathcal{H}(Z, p) \exp\left(p \sum_{i=1}^{2^d} Z_i W_i \right) \tag{2.15}$$

where

$$\mathcal{H}(Z, p) = \frac{\exp(2^d \hbar(Z) - K_0)}{I_0(2p \sum_{i=1}^{2^d} Z_i) - I_2(2p \sum_{i=1}^{2^d} Z_i)}. \tag{2.16}$$

Recursion relations for the coupling constants of a given basis can be found as described earlier, i.e. by taking the appropriate number of partial derivatives and then setting all $Z_i = 0$. As we clearly have an infinite set of different coupling constants, this would lead to an infinite number of non-trivially coupled recursion relations. Thus we again have to make a truncation of the renormalized Hamiltonian, and drop all terms in $\hbar(Z)$ in which more than a fixed number of spins interact. In order for such a truncation to be meaningful, one must first of all be sure that the relevant fixed point $\{K_n^*\}$ does indeed lie so close to the origin of all higher coupling constants that they can safely be ignored. But of course this condition is only necessary, not sufficient. The RG flow itself can at intermediate steps pass through regions of coupling-constant space where higher-order couplings are large.

3. Keeping only quadratic interactions

We first make an extremely drastic truncation down to just quadratic couplings. These are all of the form $Z_i Z_j$, with possibly $i = j$, and are illustrated graphically in figure 2 for the case $d = 2$.
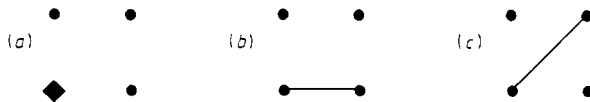

Figure 2. The 2-spin interactions in the case $d = 2$. ● stands for self-interactions of the type Z_i^2.

The decimation transformation has already projected us down to the symmetric subspace in which the couplings illustrated by figures 2(b) and (c) are equal. As we shall see shortly, this symmetric subspace is indeed stable under the LBRG transformation. Including a constant term, our initial $\hbar(Z)$ is then of the form

$$\hbar(Z) = K_0 + K_{21} \sum_i Z_i^2 + K_{22} \sum_{i<j} Z_i Z_j \qquad (i, j = 1, \dots, 2^d) \tag{3.1}$$

where it follows from the decimation transformation (2.11) that we have

$$K_0 = 0 \qquad \text{and} \qquad 2K_{21} = K_{22} = K_{nn}^2. \tag{3.2}$$

In a similar manner we establish the recursion relations from (2.15) and (2.16). This gives us

$$K_0' = 2^d K_0 + \ln \mathscr{J}_0 \qquad K_{21}' = \tfrac{1}{2} p^2 \mathscr{J}_2 \qquad K_{22}' = p^2 \mathscr{J}_{11} \qquad (3.3)$$

where $\mathscr{K}(Z, p)$ is as defined in (2.16), the integral measure is given by (2.6), and we have defined

$$\mathscr{J}_0 = \int \prod_{i=1}^{2^d} [\mathrm{d}Z_i] \mathscr{K}(Z, p) \qquad (3.4a)$$

$$\mathscr{J}_{n_1 \ldots n_\alpha} = \frac{1}{\mathscr{J}_0} \int \prod_{i=1}^{2^d} [\mathrm{d}Z_i] Z_1^{n_1} \ldots Z_\alpha^{n_\alpha} \mathscr{K}(Z, p). \qquad (3.4b)$$

We have used numerical methods to solve the RG equations (3.3), and hence obtain the RG flow in this simple truncated system. To this end, it is extremely efficient to make use of the formula

$$\int_{-1}^{1} \mathrm{d}x \sqrt{1 - x^2} f(x) = \prod_{j=1}^{n} \omega_j f(x_j) + R_n \qquad (3.5)$$

with

$$\omega_j = \frac{\pi}{n+1} \sin^2\left(\frac{j\pi}{n+1}\right) \qquad \text{and} \qquad x_j = \cos\left(\frac{j\pi}{n+1}\right) \qquad (3.6)$$

for $j = 0, \ldots, n$. Comparing with the SU(2) Haar measure of (2.6), the identity (3.5) (with $R_n \to 0$ as $n \to \infty$) is seen to be tailored for this SU(2) integration problem. In fact, we have found it far more efficient than, say, Monte Carlo integration, even in cases of relatively high dimensionality. The number of terms kept in the sum (3.5) could typically be limited to $n \sim 10$–20.

In both $d = 2$ and $d = 3$ dimensions we have found a fixed point structure and RG flow diagram as we should expect from the universality arguments connecting this model to a Z(2) invariant spin system. Starting on the original K_{nn}-axis there are two attractive fixed points, corresponding to high-temperature and low-temperature sinks. In between there is one other fixed point, attractive along a critical surface crossing the original K_{nn}-axis, repulsive in one other direction. This is the critical fixed point. This structure can be seen on a schematic flow diagram in figure 3(*a*) for a fixed value of p.

We have found this simple fixed point picture to be valid for a range of the parameter p, see figure 3(*b*). There we plot K_{22} of the three fixed points as a function of p.

The LBRG now chooses a value p^* (if it exists) such that the free energy around the fixed point is *maximized* at this value. To find this, and to compute critical exponents around the critical fixed point, we first linearize the recursion relations close to this critical point:

$$\delta K_\beta' = \sum_\alpha \left(\frac{\partial K_\beta'}{\partial K_\alpha}\right)_{K = K^*} \delta K_\alpha \equiv \mathscr{L}(p)_{\beta\alpha} \delta K_\alpha. \qquad (3.7)$$

At the critical point K^* we find that the matrix $\mathscr{L}(p)$ has two relevant eigenvalues, i.e. two eigenvalues larger than unity. One is the trivial $\lambda_0 = b^d$, where b is the rescaling factor of the RG transformation, here $b = 2$. The other, λ_1, can be related to the critical
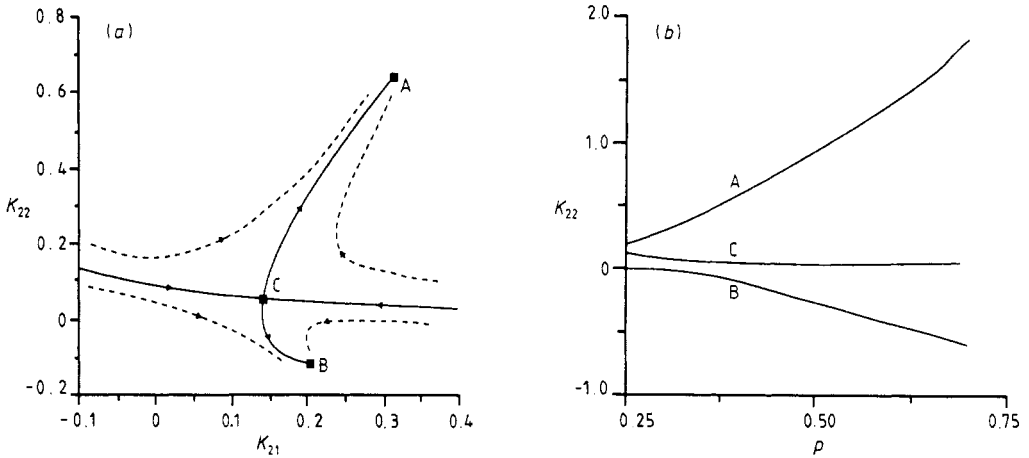
Figure 3. The case $d = 2$ and only 2-spin interactions: (a) shows a schematic flow diagram for $p = p^* = 0.416\,43$ in which the fixed point structure can be seen. A is a low-temperature sink, B a high-temperature sink and C the critical fixed point. The drawn line is the critical line; (b) shows the value of the coupling K_{22} as a function of p.

exponent α that describes the divergence of the specific heat at the critical point. The relation is (see e.g. [13])

$$\alpha = 2 - \frac{d \ln b}{\ln \lambda_1}. \tag{3.8}$$

Simultaneously we use the matrix $\mathcal{L}(p)$ to determine the value of p that gives the optimized bound on the free energy. As shown by Kadanoff *et al* [2], this is equivalent to solving the equation

$$v_0^\alpha(p) w_\alpha(p) = 0 \tag{3.9}$$

where $v_0(p)$ is the left eigenvector of $\mathcal{L}(p)$ associated with the eigenvalue λ_0, and $w(p)$ is defined by

$$w_\alpha(p) = \frac{\partial K^\alpha}{\partial p}\bigg|_{K=K^*}. \tag{3.10}$$

With these relations it is now straightforward to extract the optimal value of p, p^*, the corresponding fixed point $\{K_\beta^*\}$, and the critical exponent α. It is now also straightforward to find numerically the point at which the critical hypersurface crosses the K_{22}-axis. By using the relation (3.2) this then gives us for $d = 2$ the critical coupling K_{nn}^c of the original model. The results are shown in table 1, together with results obtained for the magnetization exponent β, the determination of which will be discussed next.

Since the magnetization exponent β refers to the response to a 'magnetic field' coupling directly to $W(x)$, it is not surprising that in order to determine β we need to add such a term $h \Sigma_x W(x)$ to the model [13]. The Hamiltonian is now no longer symmetric under the Z(2) transformation, and we need to include all *odd* operators as well, up to the given order[†]. In this first crude approximation this simply corresponds

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[†] We define *even* and *odd* operators according to their transformation properties under Z(2).

Table 1. The optimal values of the variational parameter p, in case of truncation down to 2-spin interactions, both in $d = 2$ and $d = 3$. For this value of p, the couplings, K_{21} and K_{22}, the critical exponents α and β, and, for $d = 2$, the critical temperature K_{nn}^c are displayed. Those values are compared with other known results: (1) exact values for the Ising model, (2) values obtained from series expansion for the Ising model and (3) values obtained with a Migdal–Kadanoff approximation for the SU(2) model.

d	p^*	K_{21}	K_{22}	α	β	K_{nn}^c
2	0.416 43	0.145 4	0.052 72	0.892	−0.0600	0.279
				$(0)^1$	$(0.125)^1$	$(0.2138)^3$
3	0.1 / 8 7	0.022 18	0.017 80	0.266	0.0465	
				$(0.08 \pm 0.04)^2$	$(0.3125 \pm 0.005)^2$	

to the magnetic field operator itself, i.e. a term of the form $K_1 \Sigma_i Z_i$. This leads, of course, to new sets of recursion relations, in which we now also have to include all one-hypercube expectation values of odd operators, as the Hamiltonian is no longer even under $Z(2)$. This new set of recursion relations is:

$$K_0' = 2^d K_0 + \ln \mathscr{J}_0 \qquad K_1' = p\mathscr{J}_1$$
$$K_{21}' = \tfrac{1}{2}p^2(\mathscr{J}_2 - \mathscr{J}_1^2) \qquad K_{22}' = p^2(\mathscr{J}_{11} - \mathscr{J}_1^2). \qquad (3.11)$$

But since the critical fixed point we are interested in has $K_1^* = 0$, the value of p^* is unchanged.

The matrix $\mathscr{L}(p)$ in this case has three eigenvalues larger than unity. Two of these are just the λ_0 and λ_1 found earlier, and the third, λ_2, can be related to the magnetic index β through the relation [13]

$$\beta = \frac{1}{\ln \lambda_1}(d \ln b - \ln \lambda_2). \qquad (3.12)$$

The results, summarized in table 1, are compared with both expectations based on $Z(2)$ universality and with earlier results obtained for the same model in the Migdal–Kadanoff RG approximation [14]. As one should expect for such a crude approximation to the full untruncated LBRG method, results do not compare favourably at all with either set of numbers†. Only the fixed point structure has come out as expected for an Ising-like critical point.

4. Including higher orders

After what should really only be considered a warm-up exercise, we next want to systematically study how the LBRG changes as higher-order terms are included in the truncated Hamiltonian $\hbar(Z)$. To begin, we display in figure 4 the possible new interactions when we consider the $Z(2)$ even sector of 4-spin interactions. In the symmetric case interaction (b) equals (c), (d) equals (e) and (f) equals (g). The

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† A negative index β from LBRG seems also to come out of the ordinary two-dimensional Ising model on a triangular lattice [15].
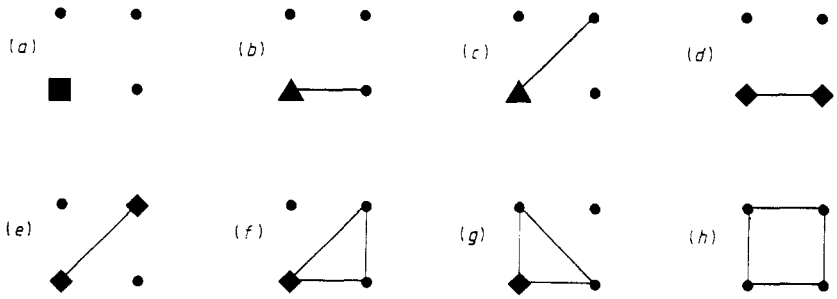
Figure 4. The 4-spin interactions in the case $d = 2$. Here $\blacklozenge$ stands for self-interaction of the type Z_i^2, $\blacktriangle$ for Z_i^3 and $\blacksquare$ for Z_i^4.

Hamiltonian now takes the form

$$\hbar(Z) = K_0 + K_{21} \sum_i Z_i^2 + K_{22} \sum_{i<j} Z_i Z_j + K_{41} \sum_i Z_i^4 + K_{42} \sum_{i \neq j} Z_i^3 Z_j + K_{43} \sum_{i<j} Z_i^2 Z_j^2$$

$$+ K_{44} \sum_{\substack{i \neq j < k \\ i \neq k}} Z_i^2 Z_j Z_k + K_{45} \sum_{i<j<k<l} Z_i Z_j Z_k Z_l. \tag{4.1}$$

Using the same technique as described in section 3, we find that the decimation equations read

$$K_0 = 0 \qquad 2K_{21} = K_{22} = K_{nn}^2$$

$$24 K_{41} = 6 K_{42} = 4 K_{43} = 2 K_{44} = K_{45} = -K_{nn}^4. \tag{4.2}$$

Finally, from (2.15) we find the RG equations of this one-hypercube approximation to be

$$K_0' = 2^d K_0 + \ln \mathcal{J}_0 \qquad\qquad K_{42}' = p^4 (\tfrac{1}{6}\mathcal{J}_{31} - \tfrac{1}{2}\mathcal{J}_2 \mathcal{J}_{11})$$

$$K_{21}' = \tfrac{1}{2} p^2 \mathcal{J}_2 \qquad\qquad K_{43}' = p^4 (\tfrac{1}{4}\mathcal{J}_{22} - \tfrac{1}{4}\mathcal{J}_2^2 - \tfrac{1}{2}\mathcal{J}_{11}^2)$$

$$K_{22}' = p^2 \mathcal{J}_{11} \qquad\qquad K_{44}' = p^4 (\tfrac{1}{2}\mathcal{J}_{21} - \mathcal{J}_{11}^2 - \tfrac{1}{2}\mathcal{J}_2 \mathcal{J}_{11}) \tag{4.3}$$

$$K_{41}' = p^4 (\tfrac{1}{24}\mathcal{J}_4 - \tfrac{1}{8}\mathcal{J}_2^2) \qquad\qquad K_{45}' = p^4 (\mathcal{J}_{1111} - 3\mathcal{J}_{11}^2)$$

using the definitions in (3.4).

We have solved the recursion relation (4.3) numerically for positive values of p (the equations are symmetric under exchange $p \leftrightarrow -p$) in the case $d = 2$, and we have found for almost all values of p that they have only one solution. Only in an *extremely* short interval, $0.383 < p < 0.395$, will there exist two attractive fixed points and a critical point between them (see figure 5(a)). As one might expect, the relevant eigenvalue starts out at the beginning of the interval with $\lambda = 1$, and ends at the end of the interval again with the value $\lambda = 1$. In between it also remains very close to unity; (in contrast to the Ising value of $\lambda = 2$). The free energy cannot be maximized anywhere inside this narrow interval, hence it is maximized on the boundary where $\lambda = 1$. The critical surface crosses the K_{22}-axis at a negative value; it is an antiferromagnetic fixed point. Due to the decimation transformation (4.2) it can, however, never be reached starting
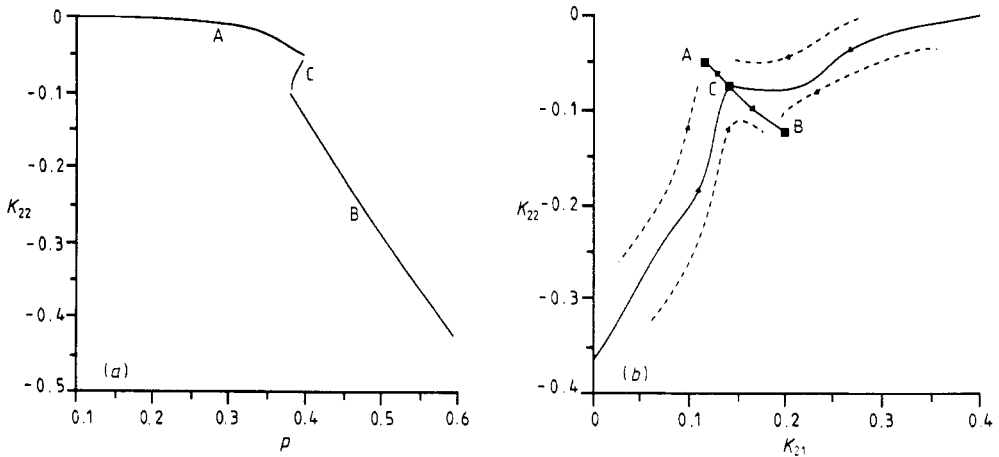
Figure 5. The case $d = 2$ and truncation to 4-spin interactions. In (a) we display the coupling constant K_{22} as a function of p for those fixed points we found. One can see that only three fixed points exist simultaneously in an extremely short interval of p, and we have a critical antiferromagnetic fixed point (C) there. In (b) we show a schematic flow diagram, projected onto the K_{21}-K_{22} plane, for one value of p in the interval, $p = 0.39$. Also shown is the critical hypersurface.

from the original model (1.2). In fact, this fixed point on a very narrow p-interval is probably completely spurious. What happens can be seen in more detail in figure 6, where we display flow diagrams for two values of p, one just before the second attractive fixed point is created, and one just after the first one disappears. By comparing those diagrams with figure 5(b) one sees a 'remnant' of the fixed points even when they have disappeared as the flow is forced towards those regions, and only bends off towards the true fixed point extremely slowly. By comparing with figure 4(a) we also note that both attractive fixed points lie very close to where the high-temperature fixed point is positioned in the case of a truncation to just 2-spin interactions. There is also an attractive area, but no fixed point, close to where the low-temperature fixed point used to be. It is as if adding the 4-spin operators made the low-temperature fixed point unstable[†]. As this is a very surprising result, we have performed a number of tests. For example, we have increased the numerical accuracy of our integration procedure (i.e. increased n in (3.5)), but have found that our results are already completely stable. We have tested directly if, starting with different values of the original coupling K_{nn}, we always flow into the same fixed point, and have found that this is indeed the case.

In the case of $d = 3$, we find similar results, with no critical hypersurface crossing the K_{22}-axis.

The conclusion is that the simple fixed-point structure of the lowest-order approximation appears to be unstable towards higher-order terms in the expansion of the one-hypercube Hamiltonian $\hbar(Z)$. Although this is a disturbing result, we must keep in mind that the method was not originally designed for such a system, and that our

[†] The fact that one nevertheless finds an attractive region around the position where one would expect the low-temperature fixed point forced us to investigate more closely whether numerical uncertainties could be the cause of this behaviour. However, we have not found any evidence substantiating this.
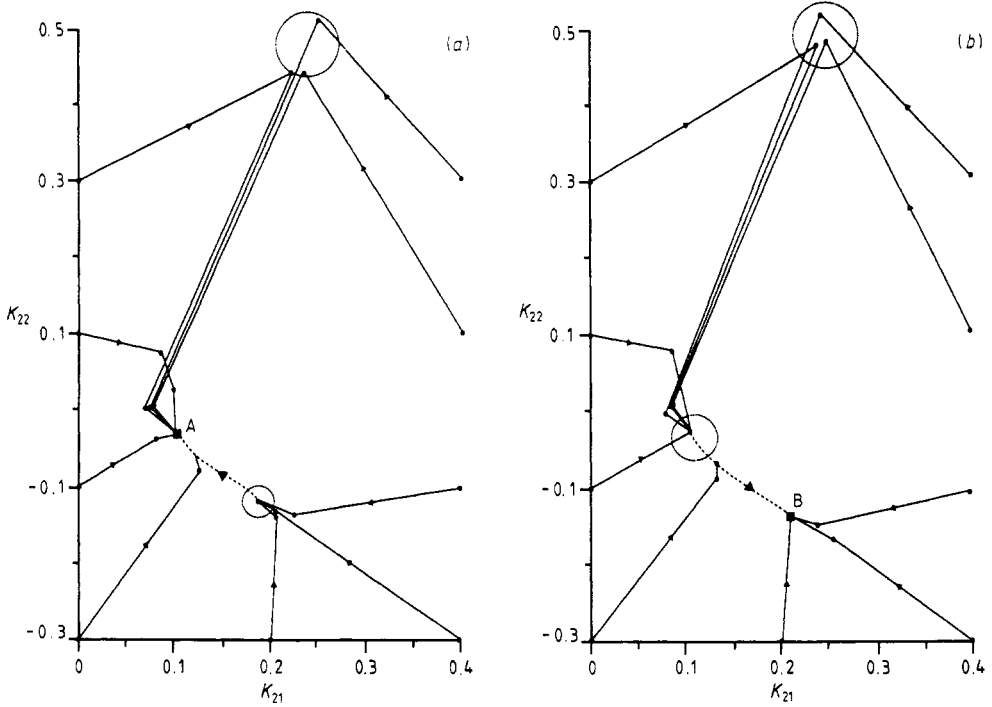
Figure 6. The same case as figure 5. Here we have flow diagrams for two values of p; one just before the antiferromagnetic fixed point is created and one just after it is destroyed. The broken circles denote attractive areas, and in (a) we have such area where the fixed point B ought to be. There is a slow flow line (broken) from B to the attractive fixed point A. This picture is reversed in (b). Otherwise the flows are identical, and both have an attractive area where the low-temperature fixed point used to be in the approximation of 2-spin interactions. In (a), $p = 0.382$, and is (b) $p = 0.396$.

definitions contain a large amount of arbitrariness. For example, in translating Kadanoff's original method [1, 2] to this system, we have chosen the projection operator of the form (2.12). Another equally valid choice, which also could be thought of as derived from Kadanoff's prescription would be e.g. $\mathcal{P}(W, Z) = \exp[p \sum_i \mathrm{Tr}(Z_i W_i)]$, where we have indicated the trace explicitly in order to distinguish it from the definition (2.12). (In this notation (2.12) would correspond to $\mathcal{P}(W, Z) = \exp[p \sum_i \mathrm{Tr}(Z_i) \mathrm{Tr}(W_i)]$.) Interestingly, we can actually also work with a purely *imaginary* p, $p \to ip$. This still leaves the recursion relations (4.3) completely real, with basically only the modified Bessel function, $I_n(x)$, replaced by $J_n(x)$, where $J_n(x)$ are the ordinary Bessel functions. Equally remarkable, the RG equations of this new definition leads to an almost identical flow pattern with the same problems as discussed above.

Since the RG flow is so substantially altered by adding the next-order terms in our expansion, it is of interest to identify precisely which terms in (4.1) are responsible for this change. It actually turns out that certain combinations of terms destroy the flow pattern of the lowest-order approximation. In table 2 we show a typical example, here for $p = 0.4$ and $d = 2$. We have indicated how the critical surface appears and

Table 2. The table shows, in the case $d = 2$, how the existence of a critical hypersurface crossing the K_{nn}-axis depends on different combinations of interaction included in the Hamiltonian. • means that a corresponding interaction is included, y means that we found a critical hypersurface and n that we did not.

K_{21}	K_{22}	K_{41}	K_{42}	K_{43}	K_{44}	K_{45}		K_{21}	K_{22}	K_{41}	K_{42}	K_{43}	K_{44}	K_{45}	
•	•						y	•	•	•	•	•			n
•	•	•					y	•	•	•	•		•		n
•	•		•				n	•	•	•	•			•	y
•	•			•			y	•	•	•		•	•		n
•	•				•		n	•	•	•		•		•	y
•	•					•	y	•	•	•			•	•	y
•	•	•	•				n	•	•		•	•	•		n
•	•	•		•			y	•	•		•	•		•	n
•	•	•			•		n	•	•		•		•	•	n
•	•	•				•	y	•	•			•	•	•	n
•	•		•	•			n	•	•	•	•	•	•		n
•	•		•		•		n	•	•	•	•	•		•	n
•	•		•			•	n	•	•	•	•		•	•	n
•	•			•	•		n	•	•	•		•	•	•	n
•	•			•		•	y	•	•		•	•	•	•	n
•	•				•	•	y	•	•	•	•	•	•	•	n

disappears as the various combinations of 4-spin operators are added. It is clear from this example that it is not one particular term among those of (4.1) which so drastically changes the RG picture.

We also tried to add all possible 6-spin interactions to the Hamiltonian (4.1) and have calculated the corresponding recursion relations. But we have found, for the case $d = 2$, that those relations give results very similar to those where only 4-spin interactions were included.

Having seen that the expansion seems to fail in this case, one would clearly like to know if this might be a consequence of staying only inside the *symmetric* subspace of operators in which the Hamiltonian is symmetric under exchange of any two spins. We have therefore extended the above analysis to the general case in which one simply starts with (2.4) and then applies the LBRG without an initial decimation transformation. In the case of $d = 2$, and truncation down to 4-spin interactions, this leads to the interactions in figures 2 and 4, without the identifications made to bring us to the symmetric subspace. The corresponding Hamiltonian is

$$h(Z) = K_0 + K_{21} \sum_{i=1}^{4} Z_i^2 + K_{22} \sum_{i=1}^{4} Z_i Z_{i+1} + K_{23} \sum_{i=1}^{2} Z_i Z_{i+2} + K_{41} \sum_{i=1}^{4} Z_i^4$$

$$+ K_{42} \sum_{\substack{i=1 \\ j=\pm 1}}^{4} Z_i^3 Z_{i+j} + K_{43} \sum_{i=1}^{4} Z_i^3 Z_{i+2} + K_{44} \sum_{i=1}^{4} Z_i^2 Z_{i+1}^2 + K_{45} \sum_{i=1}^{2} Z_i^2 Z_{i+2}^2$$

$$+ K_{46} \sum_{\substack{i=1 \\ j=\pm 1}}^{4} Z_i^2 Z_{i+j} Z_{i+2} + K_{47} \sum_{i=1}^{4} Z_i^2 Z_{i+1} Z_{i+3} + K_{48} \prod_{i=1}^{4} Z_i. \qquad (4.4)$$

The full LBRG equations read (in the notation of (3.4))[†]

$$K'_0 = 4K_0 + \ln \mathscr{J}_0 \qquad K'_{41} = p^4(\tfrac{1}{24}\mathscr{J}_{4000} - \tfrac{1}{8}\mathscr{J}^2_{2000})$$

$$K'_{21} = \tfrac{1}{2}p^2\mathscr{J}_{2000} \qquad K'_{42} = p^4(\tfrac{1}{6}\mathscr{J}_{3100} - \tfrac{1}{2}\mathscr{J}_{2000}\mathscr{J}_{1100})$$

$$K'_{22} = p^2\mathscr{J}_{1100} \qquad K'_{43} = p^4(\tfrac{1}{6}\mathscr{J}_{3010} - \tfrac{1}{2}\mathscr{J}_{2000}\mathscr{J}_{1010})$$

$$K'_{23} = p^2\mathscr{J}_{1010} \qquad K'_{44} = p^4(\tfrac{1}{4}\mathscr{J}_{2200} - \tfrac{1}{4}\mathscr{J}^2_{2000} - \tfrac{1}{2}\mathscr{J}^2_{1100})$$

$$K'_{45} = p^4(\tfrac{1}{4}\mathscr{J}_{2020} - \tfrac{1}{4}\mathscr{J}^2_{2000} - \tfrac{1}{2}\mathscr{J}^2_{1010}) \tag{4.5}$$

$$K'_{46} = p^4(\tfrac{1}{2}\mathscr{J}_{2110} - \mathscr{J}_{1100}\mathscr{J}_{1010} - \tfrac{1}{2}\mathscr{J}_{2000}\mathscr{J}_{1100})$$

$$K'_{47} = p^4(\tfrac{1}{2}\mathscr{J}_{2101} - \mathscr{J}^2_{1100} - \tfrac{1}{2}\mathscr{J}_{2000}\mathscr{J}_{1010})$$

$$K'_{48} = p^4(\mathscr{J}_{1111} - 2\mathscr{J}^2_{1100} - \mathscr{J}^2_{1010}).$$

The RG flow pattern of this general set of equations is far more complex than that of the symmetric LBRG. For example choosing a sufficiently large variational parameter p, one easily find regions of coupling-constant space at which the RG flow does not converge on stable fixed points, but rather enters apparently unending *limit cycles*. We show an example of this in figure 7. The number of fixed points is bigger than that of the symmetric subspace, since of course we already have the fixed points of the symmetric subspace. But we have not been able to find any critical point in a critical hypersurface crossing the K_{nn}-axis. So again the assumption of a rapidly converging expansion of simple local spin interactions for this SU(2) model does not seem to be justified within the framework of the LBRG.
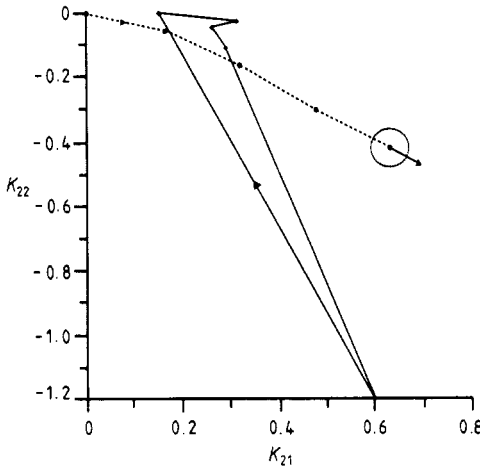
Figure 7. A projection, onto the K_{21}-K_{22} plane, of a flow for the non-symmetric case which shows the existence of limit cycles. This is for $d = 2$, $p = 0.6$ and truncation to 4-spin interactions. We start with all couplings equal to zero, then follow the broken line to an attractive area (broken circle). As there is no fixed point there the flow starts to jump abruptly around. After about 50 iterations it enters an apparently stable limit cycle (the full lines).

[†] Here the spins are numbered rotating on the square. As the Hamiltonian, and the integrals in (3.4), are no longer invariant under arbitrary permutation of the spins, but only those belonging to opposite corners, we have written out all the indices of the integrals to avoid confusion.

5. Conclusions

The object of this paper was to investigate whether Kadanoff's LBRG transformation can be applied successfully to spin models with an infinite basis of operators on one hypercube. The SU(2) example, chosen because it is of interest for the study of deconfinement phase transitions in strongly interacting matter, should not be atypical in any respect. We have found that the most drastic truncation of the operator basis to just quadratic terms (nearest-neighbour interactions and self-interactions) leads to a fixed point structure compatible with other approaches. The critical indices of this first, and very crude, approximation are not close to Ising values (as expected from universality arguments), but this is hardly surprising. The problem lies in the fact that this operator truncation appears to be inconsistent, since higher-order terms can completely destroy this picture. Curiously, if instead of directly integrating the basic building blocks (3.4) one performs an analytical expansion of these integrals, one finds a simple set of coupled polynomials recursion relations. These equations, which may be viewed as the direct generalization of the method of [6] to this model, do show the anticipated fixed-point structure. However, the expansion at the optimized choice p^*, which is based on fewer approximations, is indeed even *less* justified for this model than the expansions discussed in detail here. So it is difficult to regard this as more than an accidental result. At least, if one attempts to improve this method in the manner suggested here, one runs into difficulties. We find it challenging to see if alternative expansions, combined with the LBRG scheme, can be found to overcome this problem.

Acknowledgment

References

[1] Kadanoff L P 1975 *Phys. Rev. Lett.* **34** 1005
[2] Kadanoff L P, Houghton A and Yalabík M C 1976 *J. Stat. Phys.* **14** 171
[3] Burkhardt T W 1982 *Real-Space Renormalization* ed T W Burkhardt and J M J van Leeuwen (Berlin: Springer) pp 33-56
[4] Knops H J F 1978 *Physica* **93A** 427
[5] Callaway D J E 1989 *Phys. Rev.* D **39** 612
[6] Droz M and Malaspinas A 1978 *J. Phys. C: Solid State Phys.* **11** 2729
[7] Burkhardt T W and Kinzel W 1979 *Phys. Rev.* B **20** 4730
[8] Green F and Karsch F 1984 *Nucl. Phys.* B **238** 297
[9] Polyakov A M 1978 *Phys. Lett.* **72B** 477
    Susskind L 1979 *Phys. Rev.* D **20** 2610
[10] Svetitsky B and Yaffe L G 1982 *Nucl. Phys.* B **210** 423
[11] Burkhardt T W 1976 *Phys. Rev.* B **13** 3187
[12] Kadanoff L P and Houghton A 1975 *Phys. Rev.* B **11** 377
[13] Niemeijer Th and van Leeuwen J M J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic)
[14] Damgaard P H and Patkos A 1986 *Nucl. Phys.* B **272** 701
[15] Jan N and Glazer A M 1978 *Physica* **91A** 461